

# Exact Groups, Induced Ideals, and Fell Bundles

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**ABSTRACT.** Given a  $C^*$ -algebra  $B$  which is graded over a discrete group  $G$  we consider ideals of  $B$  which are invariant under the projections onto each of the grading subspaces. If  $G$  is exact and the standard conditional expectation of  $B$  is faithful we show that all such ideals are induced, i.e. are generated by their intersection with the unit fiber algebra. This result is derived from a generalization to the context of Fell bundles of a Theorem by Kirchberg and Wasserman according to which a group is exact if and only if its reduced  $C^*$ -algebra is exact.

## 1. Introduction.

Recall from [KW] that a locally compact group  $G$  is said to be exact if the operation of taking reduced crossed products by  $G$  preserves short exact sequences of  $C^*$ -algebras. More specifically  $G$  is exact if, given any short exact sequence

$$0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$$

of  $C^*$ -algebras carrying actions of  $G$ , and such that all maps are covariant, the sequence resulting from taking the reduced crossed product by  $G$  remains exact. In [KW: 5.2] Kirchberg and Wasserman showed that a discrete group  $G$  is exact in the above sense if and only if its reduced  $C^*$ -algebra  $C_r^*(G)$  is an exact  $C^*$ -algebra (i.e. if taking the minimal tensor product with  $C_r^*(G)$  preserves short exact sequences of  $C^*$ -algebras [W]).

It has long been realized that crossed products may be viewed as a special case of cross-sectional  $C^*$ -algebras in the context of Fell bundles (see [FD] for a comprehensive treatment of the theory of Fell bundles, also referred to as  $C^*$ -algebraic bundles) and hence it is natural to ask for a generalization of Kirchberg and Wasserman's result to this wider context. It is the purpose of the present work to prove precisely such a generalization.

As a byproduct we obtain a result about induced ideals on  $C^*$ -algebras which are graded over exact groups along the lines of similar results obtained by Strătilă and Voiculescu on AF-algebras [SV], by Nica on quasi-lattice ordered groups [N], and by the author on Fell bundles [E2]. See also Remark 3.2 in [EL].

We say that a  $C^*$ -algebra  $B$  is graded over a discrete group  $G$  if  $B$  is the closure of a direct sum of the form  $\bigoplus_{t \in G} B_t$  where the  $B_t$ 's are closed linear subspaces satisfying  $B_t B_s \subseteq B_{ts}$  and  $B_t^* = B_{t^{-1}}$ . In all of the interesting examples there is a conditional expectation of  $B$  onto  $B_e$  vanishing on the other  $B_t$ 's. By [E2: 3.5] the existence of such a map implies that the canonical projections

$$F_t : \bigoplus_{t \in G} B_t \rightarrow B_t$$

extend to bounded linear maps on  $B$ . For an element  $b \in B$  the *coefficients*  $F_t(b)$  often play the role of Fourier coefficients and much of the harmonic analysis regarding the convergence of Fourier series carry over to this situation. It is therefore natural to make the hypothesis that such a conditional expectation exists and in such a case we say (cf. [E2: Definition 3.4]) that the grading is a *topological grading*.

We say that an ideal  $\mathcal{I}$  in  $B$  is *invariant* if  $F_t(\mathcal{I}) \subseteq \mathcal{I}$  for all  $t$  in  $G$  (the reason for this terminology is that when  $G$  is abelian this is equivalent to  $\mathcal{I}$  being invariant under the dual action, that is the action of the Pontryagin dual  $\widehat{G}$  on  $B$  given by  $\alpha_x(b_t) = \langle x, t \rangle b_t$  for all  $t \in G$ ,  $x \in \widehat{G}$ , and  $b_t \in B_t$  [E1: 5.1]).

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Following [E2: 3.10] we say that  $\mathcal{I}$  is an *induced* ideal when  $\mathcal{I} = \langle \mathcal{I} \cap B_e \rangle$ . If  $G$  is an amenable group it is possible to show that every invariant ideal is induced by showing that any element  $b \in \mathcal{I}$  can be recovered from its Fourier coefficients, if not by summing its Fourier series, at least by a Cesaro means argument [E2: 4.9 and 4.10].

On the other hand, for any non-amenable group one can produce an example of an invariant ideal which is not induced: just take the kernel of the regular representation of  $C^*(G)$ . This seems to indicate that one should concentrate on “reduced” gradings, meaning topological gradings whose conditional expectation is faithful. Nevertheless, given that non-exact discrete groups exist (see [O], [G]), even under the hypothesis that the grading is reduced, one may construct an example of an invariant ideal which is not induced, as is done at the end of Section 4 in [E2].

In the main application of the present work we deduce, as a consequence of our generalization of Kirchberg and Wasserman’s result, that invariant ideals are necessarily induced whenever the base group is exact and the grading is reduced.

Last but not least we would like to acknowledge a few short but fruitful discussions with Simon Wasserman and Iain Raeburn from where some of the techniques used in this work evolved.

## 2. Ideals in Fell Bundles.

Throughout this work we shall let  $G$  be a discrete group and  $\mathcal{B} = \{B_t\}_{t \in G}$  be a Fell bundle over  $G$ .

In this section we will briefly outline the main aspects of the theory of ideals in Fell bundles which we will need in the sequel. We shall leave out most proofs since they are always easy generalizations of the corresponding ones for  $C^*$ -algebras.

**2.1. Definition.** An *ideal* in  $\mathcal{B}$  is a collection  $\mathcal{J} = \{J_t\}_{t \in G}$ , where each  $J_t$  is a closed subspace of  $B_t$ , such that for all  $s, t \in G$  one has that  $B_s J_t \subseteq J_{st}$  and  $J_s B_t \subseteq J_{st}$ .

For example, if  $J$  is an ideal in the cross-sectional  $C^*$ -algebra  $C^*(\mathcal{B})$  then, defining  $J_t = J \cap B_t$ , one has that  $\{J_t\}_{t \in G}$  is an ideal in  $\mathcal{B}$ .

Given an ideal  $\mathcal{J}$  of  $\mathcal{B}$  one clearly has that  $J_e$  ( $e$  denoting the unit element in  $G$ ) is an ideal of  $B_e$ . If  $\{u_i\}$  is an approximate identity for  $J_e$  it is easy to show that for each  $b$  in each  $J_t$  one has that  $\lim_i u_i b = \lim_i b u_i = b$ . As in the case of  $C^*$ -algebras [D: 1.8.2] it follows that  $\mathcal{J}$  is self-adjoint in the sense that  $(J_t)^* = J_{t^{-1}}$ . One then has that  $\mathcal{J}$  is a Fell bundle in its own right.

Consider, for each  $t$  in  $G$ , the quotient space  $B_t/J_t$ . It is clear that the operations of  $\mathcal{B}$  drop to the quotient giving a multiplication operation

$$\cdot : \frac{B_s}{J_s} \times \frac{B_t}{J_t} \rightarrow \frac{B_{st}}{J_{st}}$$

and an involution

$$* : \frac{B_t}{J_t} \rightarrow \frac{B_{t^{-1}}}{J_{t^{-1}}}.$$

With the quotient topology one has that the collection  $\mathcal{B}/\mathcal{J} := \{B_t/J_t\}_{t \in G}$  is then a Fell bundle (the proof of the  $C^*$ -identity  $\|b^*b\| = \|b\|^2$ , for  $b \in B_t/J_t$ , is perhaps the only slightly nontrivial verification to be made but it again follows as in [D: 1.8.2]).

The inclusion map  $\mathcal{J} \rightarrow \mathcal{B}$  and the quotient map  $\mathcal{B} \rightarrow \mathcal{B}/\mathcal{J}$  clearly give  $*$ -homomorphisms

$$C^*(\mathcal{J}) \xrightarrow{\iota} C^*(\mathcal{B}) \quad \text{and} \quad C^*(\mathcal{B}) \xrightarrow{\kappa} C^*(\mathcal{B}/\mathcal{J}).$$

**2.2. Proposition.** *If  $\mathcal{J}$  is an ideal in  $\mathcal{B}$  then*

$$0 \rightarrow C^*(\mathcal{J}) \xrightarrow{\iota} C^*(\mathcal{B}) \xrightarrow{\kappa} C^*(\mathcal{B}/\mathcal{J}) \rightarrow 0$$

*is an exact sequence of  $C^*$ -algebras.*

*Proof.* Let  $\pi$  be a faithful non-degenerate representation of  $C^*(\mathcal{J})$  on a Hilbert space  $H$ . By [FD: VIII.9.4] we have that  $\pi|_{J_e}$  is also non-degenerate.

Let  $\{u_i\}$  be an approximate identity for  $J_e$ . It is easy to see that for each  $b_t$  in each  $B_t$  the net  $\{\pi(u_i b_t)\}_i$  converges in the strong operator topology. Denoting the limit by  $\tilde{\pi}(b_t)$  one clearly has that  $\tilde{\pi}(b_t)\pi(a)\xi = \pi(b_t a)\xi$  for all  $a \in J_e$  and  $\xi \in H$ , which in turn can be used to prove that  $\tilde{\pi}$  is a \*-representation of  $\mathcal{B}$  in the sense of [FD: VIII.9]. Denote also by  $\tilde{\pi}$  the integrated form of  $\tilde{\pi}$  [FD: VIII.11.6] which is then a representation of  $C^*(\mathcal{B})$  extending  $\pi$ . Given  $a \in C^*(\mathcal{J})$  we therefore have that

$$\|a\| = \|\pi(a)\| = \|\tilde{\pi}(\iota(a))\| \leq \|\iota(a)\|$$

thus proving that  $\iota$  is one-to-one.

The composition  $\kappa \circ \iota$  clearly vanishes and hence the range of  $\iota$  is contained in  $\text{Ker}(\kappa)$ . In order to prove the reverse containment observe that, for each  $t \in G$ , the composition

$$B_t \rightarrow C^*(\mathcal{B}) \xrightarrow{q} C^*(\mathcal{B})/C^*(\mathcal{J}),$$

where the leftmost arrow is the canonical inclusion of  $B_t$  in  $C^*(\mathcal{B})$  and  $q$  is the canonical quotient map, vanishes on  $J_t$  and hence drops to the quotient yielding a map

$$j_t : B_t/J_t \rightarrow C^*(\mathcal{B})/C^*(\mathcal{J}).$$

Together the collection of maps  $\{j_t\}_{t \in G}$  forms a representation of  $\mathcal{B}/\mathcal{J}$  into  $C^*(\mathcal{B})/C^*(\mathcal{J})$  which in turn gives rise to an integrated representation, namely a \*-homomorphism

$$\psi : C^*(\mathcal{B}/\mathcal{J}) \rightarrow C^*(\mathcal{B})/C^*(\mathcal{J}),$$

whose restriction to  $B_t/J_t$  coincides with  $j_t$ . One may now easily show that  $\psi \circ \kappa$  is the canonical quotient map  $q : C^*(\mathcal{B}) \rightarrow C^*(\mathcal{B})/C^*(\mathcal{J})$ . Thus, if  $a \in \text{Ker}(\kappa)$  we have that  $0 = \psi(\kappa(a)) = q(a)$  and hence  $a \in C^*(\mathcal{J})$  as desired. This shows that our sequence is exact at the middle. Exactness at  $C^*(\mathcal{B}/\mathcal{J})$ , namely the fact that  $\kappa$  is onto, is left as an easy exercise.  $\square$

### 3. The absorption property of the left regular representation.

As before we will let  $\mathcal{B} = \{B_t\}_{t \in G}$  be a Fell bundle over the discrete group  $G$ . In this section we aim to prove that the tensor product (in a sense to be made precise below) of any representation of  $C^*(\mathcal{B})$  with the left regular representation of  $G$  leads to a representation which factors through the reduced cross-sectional C\*-algebra  $C_r^*(\mathcal{B})$ .

We refer the reader to [E2] for the basic theory of reduced cross-sectional C\*-algebras of Fell bundles but let us nevertheless recall a couple of facts from [E2] which will be crucial in what follows:

**3.1. Proposition.** *There exists a \*-homomorphism  $\Lambda : C^*(\mathcal{B}) \rightarrow C_r^*(\mathcal{B})$  which is the identity on each  $B_t$  and whose kernel is given by*

$$\text{Ker}(\Lambda) = \left\{ b \in C^*(\mathcal{B}) : E(b^*b) = 0 \right\},$$

where  $E$  is the standard conditional expectation from  $C^*(\mathcal{B})$  to  $B_e$ .

*Proof.* See the paragraph immediately after [E2: 2.3] as well as [E2: 3.6].  $\square$

Let  $\Pi$  be any representation of  $C^*(\mathcal{B})$  on a Hilbert space  $H$ , considered fixed for the time being. The restriction of  $\Pi$  to the disjoint union of the  $B_t$ 's is easily seen to form a representation of  $\mathcal{B}$  which we shall denote by  $\pi$ . Consider the representation  $\pi \otimes \lambda$  of  $\mathcal{B}$  on  $H \otimes \ell_2(G)$  given by

$$(\pi \otimes \lambda)(b_t) = \pi(b_t) \otimes \lambda_t,$$

whenever  $t \in G$  and  $b_t \in B_t$  (as usual the “ $\lambda$ ” in the right hand side above refers to the left regular representation of  $G$ ).

**3.2. Definition.** We shall denote by  $\Pi \otimes \Lambda$  the representation of  $C^*(\mathcal{B})$  obtained by integrating  $\pi \otimes \lambda$ .

Denote by  $\{\delta_t\}_{t \in G}$  the canonical orthonormal basis of  $\ell_2(G)$  and for each  $t$  let  $P_t$  be the orthogonal projection from  $H \otimes \ell_2(G)$  onto  $H \otimes \delta_t$ . In what follows we shall make use of the *compression operator*

$$F_t : \mathcal{B}(H \otimes \ell_2(G)) \ni T \longmapsto P_t T P_t \in \mathcal{B}(H \otimes \delta_t).$$

**3.3. Lemma.** If  $T$  is a nonnegative operator belonging to the range of  $\Pi \otimes \Lambda$  and  $F_e(T) = 0$  then  $T = 0$ .

*Proof.* Denote by  $\rho$  the right regular representation of  $G$  on  $\ell_2(G)$ , so that  $\rho_t(\delta_s) = \delta_{st^{-1}}$  for  $t, s \in G$ . Observe that  $1 \otimes \rho_t$  commutes with the range of  $\Pi \otimes \Lambda$  while

$$(1 \otimes \rho_{t^{-1}})P_s(1 \otimes \rho_t) = P_{st},$$

for all  $s$ . So, for any  $T$  in the range of  $\Pi \otimes \Lambda$ , we obtain

$$(1 \otimes \rho_{t^{-1}})F_s(T)(1 \otimes \rho_t) = F_{st}(T).$$

Assuming that  $F_e(T) = 0$  we then conclude that  $F_t(T) = 0$  for all  $t$  in  $G$  which therefore says that the diagonal of  $T$  vanishes relative to the decomposition

$$H \otimes \ell_2(G) = \bigoplus_{t \in G} H \otimes \delta_t.$$

If we also assume that  $T \geq 0$  this now implies that  $T = 0$ .  $\square$

The following is the main result of this section:

**3.4. Theorem.** Given any representation  $\Pi$  of  $C^*(\mathcal{B})$  on a Hilbert space  $H$  there exists a representation  $\Pi'$  of  $C_r^*(\mathcal{B})$  on  $H \otimes \ell_2(G)$  such that the diagram

$$\begin{array}{ccc} C^*(\mathcal{B}) & \xrightarrow{\Pi \otimes \Lambda} & \mathcal{B}(H \otimes \ell_2(G)) \\ \Lambda \searrow & & \nearrow \Pi' \\ & C_r^*(\mathcal{B}) & \end{array}$$

commutes. If moreover  $\Pi$  is faithful on  $B_e$  then  $\Pi'$  is faithful on  $C_r^*(\mathcal{B})$ .

*Proof.* Identifying  $H \otimes \delta_e$  and  $H$  in the obvious fashion we will think of the compression operator  $F_e$  defined above as taking values in  $H$ . Observe that for every  $b_t$  in each  $B_t$  we have

$$F(\Pi \otimes \Lambda(b_t)) = F(\pi(b_t) \otimes \lambda_t) = \begin{cases} \pi(b_e) & \text{if } t = e, \\ 0 & \text{if } t \neq e. \end{cases}$$

It follows that

$$F \circ (\Pi \otimes \Lambda) = \pi \circ E,$$

where  $E$  is the standard conditional expectation from  $C^*(\mathcal{B})$  to  $B_e$  (see [E2: 2.9]). Given  $b \in C^*(\mathcal{B})$  we then have by Lemma (3.3) that

$$(\Pi \otimes \Lambda)(b) = 0 \Leftrightarrow F((\Pi \otimes \Lambda)(b^*b)) = 0 \Leftrightarrow \pi(E(b^*b)) = 0. \quad (\dagger)$$

This implies that  $\Pi \otimes \Lambda$  vanishes on the kernel of  $\Lambda$  (see (3.1)) so that  $\Pi \otimes \Lambda$  indeed factors through  $C_r^*(\mathcal{B})$ . If we moreover assume that  $\Pi$  is faithful on  $B_e$  we have by  $(\dagger)$  that the kernel of  $\Lambda$  is precisely equal to the kernel of  $\Pi \otimes \Lambda$  and hence that  $\Pi'$  is one-to-one.  $\square$

Our next result is an important consequence of the above result.

**3.5. Corollary.** *There exists an injective \*-homomorphism*

$$\Phi : C_r^*(\mathcal{B}) \rightarrow C^*(\mathcal{B}) \otimes C_r^*(G)$$

(here and elsewhere  $\otimes$  denotes the minimal tensor product of  $C^*$ -algebras) such that  $\Phi(b_t) = b_t \otimes \lambda_t$  for every  $b_t$  in each  $B_t$ .

*Proof.* Pick a faithful representation  $\Pi$  of  $C^*(\mathcal{B})$  on a Hilbert space  $H$ . Then the representation  $\Pi'$  given by (3.4) provides the desired map.  $\square$

A last technical result is in order before we embark on our main result.

**3.6. Lemma.** *There exists a bounded linear map*

$$\Psi : C^*(\mathcal{B}) \otimes C_r^*(G) \rightarrow C_r^*(\mathcal{B})$$

such that for every  $t, s \in G$  and every  $b_t \in B_t$  one has that  $\Psi(b_t \otimes \lambda_s) = \begin{cases} b_t & \text{if } t = s, \\ 0 & \text{if } t \neq s. \end{cases}$

*Proof.* Let  $\Pi$  be a faithful representation of  $C^*(\mathcal{B})$  on a Hilbert space  $H$  so that  $\Pi \otimes \Lambda$  is a representation of  $C^*(\mathcal{B})$  on  $H \otimes \ell_2(G)$ . Denote by  $\lambda$  the identical representation of  $C_r^*(G)$  on  $\ell_2(G)$  so that  $(\Pi \otimes \Lambda) \otimes \lambda$  is a representation of  $C^*(\mathcal{B}) \otimes C_r^*(G)$  on  $H \otimes \ell_2(G) \otimes \ell_2(G)$  as seen above. Consider the subspace  $K$  of the latter given by

$$K = \bigoplus_{t \in G} H \otimes \delta_t \otimes \delta_t,$$

and let  $P$  be the orthogonal projection onto  $K$ . Also consider the map

$$\Psi : C^*(\mathcal{B}) \otimes C_r^*(G) \rightarrow \mathcal{B}(K)$$

given by  $\Psi(x) = P((\Pi \otimes \Lambda) \otimes \lambda)(x)P$ .

There is an obvious isometric isomorphism between  $K$  and  $H \otimes \ell_2(G)$  under which a vector of the form  $\xi \otimes \delta_t \otimes \delta_t$  is mapped to  $\xi \otimes \delta_t$ . If we identify  $\mathcal{B}(K)$  with  $\mathcal{B}(H \otimes \ell_2(G))$  under this map one sees that

$$\Psi(b_t \otimes \lambda_s) = \begin{cases} \Pi(b_t) \otimes \lambda_t & \text{if } t = s, \\ 0 & \text{if } t \neq s. \end{cases}$$

for every  $t, s \in G$  and every  $b_t \in B_t$ . The range of  $\Psi$  is therefore the closed linear span of the set of all  $\Pi(b_t) \otimes \lambda_t$  which happens to be also the range of  $\Pi \otimes \Lambda$ , which in turn is isomorphic to  $C_r^*(\mathcal{B})$  by (3.4). We may then view  $\Psi$  as taking values in  $C_r^*(\mathcal{B})$ , thus providing the desired map.  $\square$

#### 4. Reduced cross-sectional algebras and exact groups.

Let  $\mathcal{B}$  be a Fell bundle over the discrete group  $G$  and let  $\mathcal{J}$  be an ideal in  $\mathcal{B}$ . As seen in (2.2) there is a natural exact sequence of full cross-sectional  $C^*$ -algebras

$$0 \rightarrow C^*(\mathcal{J}) \xrightarrow{\iota} C^*(\mathcal{B}) \xrightarrow{\kappa} C^*(\mathcal{B}/\mathcal{J}) \rightarrow 0 \quad (4.1)$$

We would like to show that one gets out of this a (not necessarily exact!) sequence of reduced cross-sectional  $C^*$ -algebras. For this we need the following:

**4.2. Lemma.** *Given Fell bundles  $\mathcal{B}$  and  $\mathcal{B}'$  over the same discrete group  $G$  let  $\phi : C^*(\mathcal{B}) \rightarrow C^*(\mathcal{B}')$  be a  $*$ -homomorphism preserving fibers, i.e., such that  $\phi(B_t) \subseteq B'_t$  for all  $t$  in  $G$ . Then there exists a  $*$ -homomorphism  $\phi_r : C_r^*(\mathcal{B}) \rightarrow C_r^*(\mathcal{B}')$  such that the diagram*

$$\begin{array}{ccc} C^*(\mathcal{B}) & \xrightarrow{\phi} & C^*(\mathcal{B}') \\ \Lambda \downarrow & & \downarrow \Lambda' \\ C_r^*(\mathcal{B}) & \xrightarrow{\phi_r} & C_r^*(\mathcal{B}') \end{array}$$

commutes, where  $\Lambda$  and  $\Lambda'$  are as in (3.1) relative to  $\mathcal{B}$  and  $\mathcal{B}'$  respectively.

*Proof.* Consider the diagram

$$\begin{array}{ccc} C^*(\mathcal{B}) & \xrightarrow{\phi} & C^*(\mathcal{B}') \\ E \downarrow & & \downarrow E' \\ \mathcal{B}_e & \xrightarrow{\phi|_{B_e}} & B'_e \end{array}$$

where  $E$  and  $E'$  are the respective conditional expectations. This is clearly commutative and hence if  $b \in C^*(\mathcal{B})$  is such that  $E(b^*b) = 0$  then by (3.1) we have that  $E'(\phi(b)^*\phi(b)) = 0$ . It follows that  $\phi$  sends the kernel of  $\Lambda$  into the kernel of  $\Lambda'$  from which the existence of  $\phi_r$  follows.  $\square$

Given the exact sequence (4.1) we therefore get the (not necessarily exact!) sequence

$$0 \rightarrow C_r^*(\mathcal{J}) \xrightarrow{\iota_r} C_r^*(\mathcal{B}) \xrightarrow{\kappa_r} C_r^*(\mathcal{B}/\mathcal{J}) \rightarrow 0. \quad (4.3)$$

Our main result is precisely related to the situations in which one may guarantee such a sequence to be exact:

**4.4. Theorem.** *The following conditions on a discrete group  $G$  are equivalent:*

- (i)  $G$  is an exact group.
- (ii) For every Fell bundle  $\mathcal{B}$  over  $G$  and every ideal  $\mathcal{J}$  of  $\mathcal{B}$  one has that the sequence (4.3) is exact.

*Proof.* Consider the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & C^*(\mathcal{J}) \otimes C_r^*(G) & \xrightarrow{\iota \otimes 1} & C^*(\mathcal{B}) \otimes C_r^*(G) & \xrightarrow{\kappa \otimes 1} & C^*(\mathcal{B}/\mathcal{J}) \otimes C_r^*(G) & \rightarrow & 0 \\ & & \Phi_1 \uparrow & & \Phi_2 \uparrow & & \Phi_3 \uparrow & & \\ 0 & \rightarrow & C_r^*(\mathcal{J}) & \xrightarrow{\iota_r} & C_r^*(\mathcal{B}) & \xrightarrow{\kappa_r} & C_r^*(\mathcal{B}/\mathcal{J}) & \rightarrow & 0 \end{array}$$

where the vertical arrows are from (3.5). It is a simple matter to show that this is a commutative diagram.

Assuming that  $G$  is an exact group we have by (2.2) that the top row is exact and we must then prove that the same holds for the bottom one. The only nontrivial aspect of doing so is showing that the kernel of

$\kappa_r$  is contained in the image of  $\iota_r$  which we now set out to prove. So let  $b \in C_r^*(\mathcal{B})$  be such that  $\kappa_r(b) = 0$ . By a standard diagram chasing argument one concludes that there exists  $c \in C^*(\mathcal{J}) \otimes C_r^*(G)$  such that  $(\iota \otimes 1)(c) = \Phi_2(b)$ .

We will now consider the maps  $\Psi_1$  and  $\Psi_2$  provided by (3.6) both for  $\mathcal{J}$  and for  $\mathcal{B}$ . These are easily seen to form the commutative diagram

$$\begin{array}{ccc} C^*(\mathcal{J}) \otimes C_r^*(G) & \xrightarrow{\iota \otimes 1} & C^*(\mathcal{B}) \otimes C_r^*(G) \\ \Psi_1 \downarrow & & \Psi_2 \downarrow \\ C_r^*(\mathcal{J}) & \xrightarrow{\iota_r} & C_r^*(\mathcal{B}) \end{array}$$

It is also easy to see that the compositions  $\Psi_i \circ \Phi_i$  give the identity maps in both cases. We then have that

$$\Phi_2(\iota_r(\Psi_1(c))) = \Phi_2(\Psi_2(\iota \otimes 1(c))) = \Phi_2(\Psi_2(\Phi_2(b))) = \Phi_2(b).$$

Since  $\Phi_2$  is injective by (3.5) we conclude that  $\iota_r(\Psi_1(c)) = b$  and hence that  $b$  is in the range of  $\iota_r$ . The remaining points in the proof of the exactness of our sequence are left to the reader.

Conversely, if (ii) is assumed, let

$$0 \rightarrow J \rightarrow B \rightarrow B/J \rightarrow 0$$

be an exact sequence of  $C^*$ -algebras. If  $A$  denotes any one of the above three  $C^*$ -algebras, consider the “trivial Fell bundle”  $A \times G$  equipped with the operations  $(a, t)(b, s) = (ab, ts)$  and  $(a, t)^* = (a^*, t^{-1})$ , for  $a, b \in A$  and  $t, s \in G$ . It is then easy to see that  $J \times G$  is an ideal in  $B \times G$  and that the sequence (4.3), which by hypothesis is exact, becomes

$$0 \rightarrow J \otimes C_r^*(G) \rightarrow B \otimes C_r^*(G) \rightarrow B/J \otimes C_r^*(G) \rightarrow 0.$$

Thus  $G$  is an exact group.  $\square$

## 5. Induced and invariant ideals.

Recall from [E2:3.1] that, given a discrete group  $G$ , a  $C^*$ -algebra  $B$  is said to be *graded* (over  $G$ ) when  $B$  contains a linearly independent collection of closed subspaces  $\{B_t\}_{t \in G}$  such that for each  $t, s \in G$  one has

- (i)  $B_t^* = B_{t^{-1}}$
- (ii)  $B_t B_s \subseteq B_{ts}$
- (iii)  $B = \overline{\bigoplus_{t \in G} B_t}$ .

According to [E2:3.4] such a grading is said to be a *topological* grading if there exists a bounded linear map  $F_e : B \rightarrow B_e$  which restricts to the identity on  $B_e$  and which vanishes on all other  $B_t$ ’s. It follows that  $F_e$  is a positive contractive conditional expectation onto  $B_e$  [E2:3.3] and that there are contractive projections

$$F_t : B \rightarrow B_t$$

such that for all finite sums  $x = \sum_{t \in G} b_t$ , with  $b_t \in B_t$ , one has  $F_t(x) = b_t$  [E2:3.5]. Throughout this section we shall fix a topologically graded  $C^*$ -algebra  $B = \overline{\bigoplus_{t \in G} B_t}$ .

Given an ideal  $\mathcal{I}$  in  $B$  one may consider the following sets:

- $\mathcal{I}_1 = \langle \mathcal{I} \cap B_e \rangle$ , i.e. the ideal generated by  $\mathcal{I} \cap B_e$ ,
- $\mathcal{I}_2 = \{b \in B : F_e(b^*b) \in \mathcal{I}\}$ , and
- $\mathcal{I}_3 = \{b \in B : F_t(b) \in \mathcal{I}, \text{ for all } t \in G\}$ .

One always has  $\mathcal{I}_1 \subseteq \mathcal{I}_2 = \mathcal{I}_3$  [E2:3.9] and whenever the underlying Fell bundle  $\mathcal{B} = \{B_t\}_{t \in G}$  has the *approximation property* defined in [E2:4.4], e.g. if  $G$  is amenable [E2:4.7], one also has that  $\mathcal{I}_1 = \mathcal{I}_2$  [E2:4.10]. This should be compared to similar results by Strătilă and Voiculescu on AF-algebras [SV] and by Nica on quasi-lattice ordered groups [N]. It is our goal in this section to exhibit still another situation in which one may guarantee that  $\mathcal{I}_1 = \mathcal{I}_2$ .

**5.1. Theorem.** Let  $G$  be a discrete group and let  $B = \overline{\bigoplus_{t \in G} B_t}$  be a topologically  $G$ -graded  $C^*$ -algebra. Suppose that  $G$  is exact and that the standard conditional expectation  $F : B \rightarrow B_e$  is faithful. Then for every ideal  $\mathcal{I}$  of  $B$  one has that the sets  $\mathcal{I}_1$  and  $\mathcal{I}_2$  defined above coincide.

*Proof.* Denote by  $\mathcal{B} = \{B_t\}_{t \in G}$  the underlying Fell bundle and note that by [E2: 3.7]  $B$  is isomorphic to  $C_r^*(\mathcal{B})$ . For each  $t \in G$  let  $J_t = \mathcal{I} \cap B_t$  so that  $\mathcal{J} := \{J_t\}_{t \in G}$  is an ideal in  $\mathcal{B}$  according to (2.1). By (4.4) the sequence

$$0 \rightarrow C_r^*(\mathcal{J}) \xrightarrow{\iota_r} B \xrightarrow{\kappa_r} C_r^*(\mathcal{B}/\mathcal{J}) \rightarrow 0$$

is exact. We now claim that  $\mathcal{I}_1 = \iota_r(C_r^*(\mathcal{J}))$  and  $\mathcal{I}_2 = \text{Ker}(\kappa_r)$  from where the proof will be concluded.

For each  $t$  in  $G$  we have by [BMS: 1.7] that

$$J_t = J_t J_t^* J_t \subseteq J_e J_t \subseteq \mathcal{I}_1$$

so that  $\iota_r(C_r^*(\mathcal{J})) = \mathcal{I}_1$ . On the other hand, denoting by  $E$  the standard conditional expectation of  $C_r^*(\mathcal{B}/\mathcal{J})$  we have for any  $b \in B$  that

$$\kappa_r(b) = 0 \Leftrightarrow E(\kappa_r(b^*b)) = 0 \Leftrightarrow \kappa_r(F_e(b^*b)) = 0 \Leftrightarrow F_e(b^*b) \in \mathcal{I}.$$

□

According to [E2: 3.10] an ideal  $\mathcal{I}$  in  $B$  is said to be an *induced* ideal when  $\mathcal{I} = \langle \mathcal{I} \cap B_e \rangle$ . Let us also make the following:

**5.2. Definition.** An ideal  $\mathcal{I}$  in  $B$  is said to be *invariant* if  $F_t(\mathcal{I}) \subseteq \mathcal{I}$  for all  $t$  in  $G$ .

**5.3. Corollary.** Assume that  $G$  is exact and that the standard conditional expectation  $F_e$  is faithful on  $B$ . Then any invariant ideal is induced.

*Proof.* Let  $\mathcal{I}$  be an invariant ideal in  $B$ . Then

$$\mathcal{I} \subseteq \{b \in B : F_t(b) \in \mathcal{I}, \text{ for all } t \in G\} = \mathcal{I}_3 = \mathcal{I}_1 = \langle \mathcal{I} \cap B_e \rangle \subseteq \mathcal{I},$$

so equality holds throughout.

□

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